

N=2 Supersymmetric Integrable Hierarchies: Additional Symmetries and Darboux-Bäcklund Solutions

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We study additional non-isospectral symmetries of constrained (reduced) $N=2$ supersymmetric KP hierarchies of integrable “soliton”-like evolution equations. These symmetries are shown to form an infinite-dimensional non-Abelian superloop superalgebra. Furthermore we study the general Darboux-Bäcklund (DB) transformations (including adjoint-DB and binary DB) of $N=2$ super-KP hierarchies preserving (most of) the additional symmetries. Also we derive the explicit form of the general DB ($N=2$ “super-soliton”-like) solutions in the form of generalized Wronskian-like super-determinants.

1. Introduction

The principal importance of supersymmetric integrable systems from the point of view of theoretical physics is rooted in their intimate relevance for (multi-)matrix models in non-perturbative superstring theory ¹. Accordingly, from purely mathematical point of view, the subject of supersymmetrization of Kadomtsev-Petviashvili (KP) integrable hierarchy of “soliton”-like evolution equations and its various reductions ^{2,3,4} similarly attracted a lot of interest, especially, the supersymmetric generalizations of the inverse scattering method, bi-Hamiltonian structures, tau-functions and Sato Grassmannian approach, and the Drinfeld-Sokolov scheme.

An important role in the theory of integrable systems is being played by the notion of *additional (non-isospectral) symmetries* ⁵ (for detailed reviews of the latter subject, see refs.⁶). Additional symmetries, by definition, consist of the set of all flows on the space of the Lax operators of the pertinent integrable hierarchy which commute with the ordinary isospectral flows, the latter being generated by the complete set of commuting integrals of motion. One of the most significant manifestations of additional symmetries is the interpretation of the crucial Virasoro (and $\mathbf{W}_{1+\infty}$) constraints on partition functions of (multi-)matrix models of string theory as invariance of the τ -functions (*i.e.*, the string partition functions) under the Borel subalgebra of the Virasoro algebra of additional non-isospectral symmetries

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in the underlying integrable hierarchies (similarly for the $\mathbf{W}_{1+\infty}$ constraints).

Furthermore, the notion of additional symmetries allows to provide an alternative formulation of matrix (multi-component) KP hierarchies as ordinary scalar KP hierarchy supplemented with appropriate sets of mutually commuting additional symmetry flows (see refs.^{7,8}). In particular, one obtains an alternative formulation of various physically relevant nonlinear evolution equations in two- and higher-dimensional space-time (e.g., Davey-Stewartson and N -wave resonant systems, as well as Wess-Zumino-Novikov-Witten models of group-coset-valued fields) as additional-symmetry flows on ordinary (reduced) KP hierarchies (see refs.^{7,8,9,10}).

Finally, additional non-isospectral symmetries are expected to play an important role in the quantization of integrable systems (since the quantum state space must carry representations of the underlying additional symmetry algebra).

The first main topic of the present Letter is the study of additional non-isospectral symmetries of constrained (reduced) $N = 2$ supersymmetric KP integrable hierarchies. This is an extension of our study in ref.¹¹ of additional symmetries of $N = 1$ super-KP hierarchies. These symmetries are shown to form an infinite-dimensional non-Abelian superloop superalgebra. The second main topic is the study of general Darboux-Bäcklund (DB) transformations (including adjoint-DB and binary DB) of $N = 2$ super-KP hierarchies preserving (most of) the additional symmetries. The main result here is the derivation of the explicit form of the general DB ($N = 2$ “super-soliton”-like) solutions in the form of generalized Wronskian-like super-determinants. An extended exposition of the present results, including detailed proofs and further developments, will appear elsewhere.

2. Superspace Formulation of General $N=2$ Super-KP Hierarchy

We shall use throughout the $N = 2$ super-pseudo-differential operator calculus as in ref.¹² with the following notations:

$$\partial \equiv \frac{\partial}{\partial x} \quad , \quad \mathcal{D}_{\pm} = \frac{\partial}{\partial \theta_{\pm}} + \theta_{\pm} \frac{\partial}{\partial x} \quad ; \quad \mathcal{D}_{\pm}^2 = \partial \quad , \quad \{\mathcal{D}_+, \mathcal{D}_-\} = 0 \quad (1)$$

where (x, θ_+, θ_-) denote $N = 2$ superspace coordinates. For any $N = 2$ super-pseudo-differential operator \mathcal{A} :

$$\mathcal{A} = \mathcal{A}_+ + \mathcal{A}_- \quad , \quad \mathcal{A}_{\pm} \equiv \sum_{j \geq 0} \left(a_{\pm j}^{(0)} + a_{\pm j}^{(+)} \mathcal{D}_+ + a_{\pm j}^{(-)} \mathcal{D}_- + a_{\pm j}^{(1)} \mathcal{D}_+ \mathcal{D}_- \right) \partial^{\pm j} \quad (2)$$

the subscripts (\pm) denote its purely differential or purely pseudo-differential parts, respectively. The rules of conjugation within the super-pseudo-differential formalism are as follows (cf. first ref.⁴): $(\mathcal{A}\mathcal{B})^* = (-1)^{|A||B|} \mathcal{B}^* \mathcal{A}^*$ for any two elements with Grassmann parities $|A|$ and $|B|$; $(\partial^k)^* = (-1)^k \partial^k$, $(\mathcal{D}_{\pm}^k)^* = (-1)^{k(k+1)/2} \mathcal{D}_{\pm}^k$ and $u^* = u$ for any coefficient superfield. Also, in order to avoid confusion we shall also employ the following notations: for any super-(pseudo-)differential operator \mathcal{A}

and a superfield function f , the symbol $\mathcal{A}(f)$ or $(\mathcal{A}f)$ will indicate application (action) of \mathcal{A} on f , whereas the symbol $\mathcal{A}f$ without brackets will denote just operator product of \mathcal{A} with the zero-order (multiplication) operator f .

The general unconstrained $N = 2$ supersymmetric KP hierarchy is given by a fermionic $N = 2$ super-pseudo-differential Lax operator \mathcal{L} in terms of the bosonic $N = 2$ superspace dressing operator \mathcal{W} ¹² :

$$\mathcal{L} = \mathcal{W}\mathcal{D}_-\mathcal{W}^{-1} \quad , \quad \mathcal{W} \equiv 1 + \sum_{k=1}^{\infty} \left(w_k^{(0)} + w_k^{(+)}\mathcal{D}_+ + w_k^{(-)}\mathcal{D}_- + w_k^{(1)}\mathcal{D}_+\mathcal{D}_- \right) \partial^{-k} \quad (3)$$

obeying bosonic and fermionic evolution (isospectral) Sato equations[†]:

$$\frac{\partial}{\partial t_l} \mathcal{L} = \left[(\mathcal{L}^{2l})_+ , \mathcal{L} \right] \quad , \quad D_n^- \mathcal{L} = - \left\{ (\mathcal{L}^{2n-1})_- , \mathcal{L} \right\} \quad , \quad D_n^+ \mathcal{L} = \left\{ (\Lambda^{2n-1})_+ , \mathcal{L} \right\} \quad (4)$$

where $\Lambda = \mathcal{W}\mathcal{D}_+\mathcal{W}^{-1}$. The fermionic isospectral flows D_n^\pm in Eqs.(4) possess natural realization in terms of two infinite sets of fermionic ‘‘evolution’’ parameters $\{\rho_n^\pm\}_{n=1}^\infty$ and span $N=2$ supersymmetry algebra :

$$D_n^\pm = \frac{\partial}{\partial \rho_n^\pm} - \sum_{k=1}^{\infty} \rho_k^\pm \frac{\partial}{\partial t_{n+k-1}} \quad , \quad \{D_n^\pm , D_m^\pm\} = -2 \frac{\partial}{\partial t_{n+m-1}} \quad , \quad \{D_n^\pm , D_m^\mp\} = 0 \quad (5)$$

the rest of flow commutators being zero.

The super-Baker-Akhiezer (super-BA) and the adjoint super-BA wave functions are defined as $\psi_{BA} = \mathcal{W}(\psi_{BA}^{(0)})$ and $\psi_{BA}^* = \mathcal{W}^{*-1}(\psi_{BA}^{*(0)})$ in terms of the ‘‘free’’ super-BA functions (with η_\pm being fermionic ‘‘spectral’’ parameters):

$$\psi_{BA}^{(0)}(t, \theta_\pm, \rho^\pm; \lambda, \eta_\pm) = e^{\xi(t, \theta_\pm, \rho^\pm; \lambda, \eta_\pm)} \quad , \quad \psi_{BA}^{*(0)}(t, \theta_\pm, \rho^\pm; \lambda, \eta_\pm) = e^{-\xi(t, \theta_\pm, \rho^\pm; \lambda, \eta_\pm)} \quad (6)$$

$$\xi(t, \theta_\pm, \rho^\pm; \lambda, \eta_\pm) \equiv \sum_{l=1}^{\infty} \lambda^l t_l + \sum_{\pm} \left\{ \eta_\pm \theta_\pm + (\eta_\pm - \lambda \theta_\pm) \sum_{n=1}^{\infty} \lambda^{n-1} \rho_n^\pm \right\} \quad (7)$$

for which it holds:

$$\frac{\partial}{\partial t_k} \psi_{BA}^{(0)} = \partial^k \psi_{BA}^{(0)} \quad , \quad D_n^\pm \psi_{BA}^{(0)} = \mathcal{D}_\pm^{2n-1} \psi_{BA}^{(0)} = \partial_x^{n-1} \mathcal{D}_\pm \psi_{BA}^{(0)} \quad (8)$$

In Eqs.(6)–(7) and in what follows we employ the short-hand notions $(t) \equiv (t_1 \equiv x, t_2, t_3, \dots)$ and $(\rho^\pm) \equiv (\rho_1^\pm, \rho_2^\pm, \rho_3^\pm, \dots)$. Accordingly, (adjoint) super-BA wave functions satisfy:

$$(\mathcal{L}^2)^{(*)} \psi_{BA}^{(*)} = \lambda \psi_{BA}^{(*)} \quad , \quad \frac{\partial}{\partial t_l} \psi_{BA}^{(*)} = \pm (\mathcal{L}^{2l})_+^{(*)} (\psi_{BA}^{(*)})$$

[†]In fact, as shown in¹², there exist two additional bosonic and two additional fermionic sets of flows such that the $N=2$ super-KP hierarchy exhibits $N=4$ supersymmetry.

$$D_n^- \psi_{BA}^{(*)} = \pm (\mathcal{L}^{2n-1})_+^{(*)} (\psi_{BA}^{(*)}) \quad , \quad D_n^+ \psi_{BA}^{(*)} = \pm (\Lambda^{2n-1})_+^{(*)} (\psi_{BA}^{(*)}) \quad (9)$$

Another important object in the present formalism is the notion of $N=2$ super-eigenfunctions (SEF's) obeying the following defining equations:

$$\frac{\partial}{\partial t_l} \Phi = \mathcal{L}_+^{2l}(\Phi) \quad , \quad D_n^- \Phi = \mathcal{L}_+^{2n-1}(\Phi) \quad , \quad D_n^+ \Phi = \Lambda_+^{2n-1}(\Phi) \quad (10)$$

$$\frac{\partial}{\partial t_l} \Psi = -(\mathcal{L}^{2l})_+^*(\Psi) \quad , \quad D_n^- \Psi = -(\mathcal{L}^{2n-1})_+^*(\Psi) \quad , \quad D_n^+ \Psi = -(\Lambda^{2n-1})_+^*(\Psi) \quad (11)$$

Any (adjoint) SEF possesses unique $N=2$ supersymmetric ‘‘spectral’’ representation in terms of the (adjoint) super-BA function similar to the purely bosonic and $N=1$ supersymmetric cases^{13,14}:

$$\begin{aligned} \Phi(t, \theta_\pm, \rho^\pm) &= \int d\lambda d\eta_+ d\eta_- \varphi(\lambda, \eta_\pm) \psi_{BA}(t, \theta_\pm, \rho^\pm; \lambda, \eta_\pm) \\ \Psi(t, \theta_\pm, \rho^\pm) &= \int d\lambda d\eta_+ d\eta_- \varphi^*(\lambda, \eta_\pm) \psi_{BA}^*(t, \theta_\pm, \rho^\pm; \lambda, \eta_\pm) \end{aligned} \quad (12)$$

3. Constrained N=2 Super-KP Hierarchies

We are now interested in consistent reductions of the general (unconstrained) $N=2$ super-KP hierarchy (3)–(4). In ref.¹² the authors proposed a novel class of constrained $N=2$ super-KP hierarchies defined by $N=2$ super-Lax operators of the form $\mathcal{L} = \mathcal{D}_- + \Phi \mathcal{D}_+^{-1} \Psi$ with Φ, Ψ being in general matrix-valued superfunctions. On the other hand, we can use the same logic of the construction of constrained super-KP hierarchies in ref.¹⁴ to extend it from $N=1$ to $N=2$ super-KP case. In this way we obtain the following general class $SKP_{(M_B, M_F)}^{N=2}$ of constrained (reduced) $N=2$ super-KP hierarchies defined by $N=2$ super-Lax operators:

$$\mathcal{L} \equiv \mathcal{L}_{(M_B, M_F)} = \mathcal{D}_- + \sum_{i=1}^M \Phi_i \mathcal{D}_+^{-1} \Psi_i \quad (13)$$

Here $M \equiv M_B + M_F$ with $M_{B,F}$ being the number of bosonic/fermionic superfunctions Φ_i, Ψ_i . Also, for convenience the order of indices $i = 1, \dots, M$ in (13) is chosen such that the (adjoint) SEF's $\{\Phi_i, \Psi_i\}_{i=1}^{M_B}$ are bosonic whereas the rest $\{\Phi_{M_B+i}, \Psi_{M_B+i}\}_{i=1}^{M_F}$ are fermionic.

For later use let us also write down the explicit expression of $(\mathcal{L}^K)_-$ for arbitrary integer power K of \mathcal{L} (13)^{14,12}:

$$(\mathcal{L}^{2k})_- = \sum_{i=1}^M \sum_{s=0}^{2k-1} (-1)^{|s|} \mathcal{L}^{2k-1-s}(\Phi_i) \mathcal{D}_+^{-1} (\mathcal{L}^s)^*(\Psi_i) \quad (14)$$

$$\begin{aligned}
 (\mathcal{L}^{2k+1})_- &= \sum_{i=1}^M \sum_{s=0}^{2k} (-1)^{s|i|} \mathcal{L}^{2k-s}(\Phi_i) \mathcal{D}_+^{-1} (\mathcal{L}^s)^* (\Psi_i) \\
 &+ \sum_{i=1}^M \sum_{s=0}^{2k-1} (-1)^{s|i|+s+|i|} \mathcal{L}^{2k-1-s}(\Phi_i) \mathcal{D}_+^{-1} \mathcal{D}_- (\mathcal{L}^s)^* (\Psi_i)
 \end{aligned} \tag{15}$$

As in the case of $N=1$ constrained super-KP hierarchies¹⁴, consistency of the action of isospectral flows with the constrained form (13) of $SKP_{(M_B, M_F)}^{N=2}$ super-Lax operators requires nontrivial modification of the fermionic D_n^- flows (4). The corresponding Sato evolution equations for (13) become:

$$\begin{aligned}
 \frac{\partial}{\partial t_l} \mathcal{L} &= \left[(\mathcal{L}^{2l})_+, \mathcal{L} \right] \quad , \quad D_n^+ \mathcal{L} = \left\{ (\Lambda^{2n-1})_+, \mathcal{L} \right\} \\
 D_n^- \mathcal{L} &= - \left\{ (\mathcal{L}^{2n-1})_- - X_{2n-1}, \mathcal{L} \right\}
 \end{aligned} \tag{16}$$

where (cf. Eq.(15)):

$$(\mathcal{L}^{2n-1})_- - X_{2n-1} \equiv \sum_{i=1}^M \sum_{s=0}^{2n-2} (-1)^{s(|i|+1)} \mathcal{L}^{2n-2-s}(\Phi_i) \mathcal{D}_+^{-1} (\mathcal{L}^s)^* (\Psi_i) \tag{17}$$

Accordingly, the superfunctions Φ_i and Ψ_i entering the pseudo-differential art of \mathcal{L} (13) are (adjoint) SEF's (10)–(11) obeying *modified* defining D_n^- -flow equations:

$$D_n^- \Phi_i = \left(\mathcal{L}_+^{2n-1} + X_{2n-1} \right) (\Phi_i) - 2\mathcal{L}^{2n-1}(\Phi_i) \tag{18}$$

$$D_n^- \Psi_i = - \left((\mathcal{L}^{2n-1})_+^* + X_{2n-1}^* \right) (\Psi_i) + 2 (\mathcal{L}^{2n-1})^* (\Phi_i) \tag{19}$$

Let us point out that, similarly to the $N=1$ supersymmetric case, constrained $N=2$ super-KP hierarchies (13),(16) contain among themselves $N=2$ supersymmetric extensions of various basic bosonic integrable hierarchies such as (modified) Kortevog-de Vries, nonlinear Schrödinger (AKNS hierachies, in general), Yajima-Oikawa, coupled Boussinesq-type equations etc..

All solutions of (constrained) $N=2$ super-KP hierarchies are expressed through a single $N=2$ *super-tau function* $\tau = \tau(t, \theta_{\pm}, \rho^{\pm})$ which is related to the coefficients of the pertinent $N=2$ super-Lax operator \mathcal{L} (13) and its associate Λ as follows:

$$\begin{aligned}
 (\mathcal{L}^{2k})_{(-1)} &= \frac{\partial}{\partial t_k} \mathcal{D}_+ \ln \tau \quad , \quad (\Lambda^{2n-1})_{(-1)} = D_n^+ \mathcal{D}_+ \ln \tau \\
 (\mathcal{L}^{2n-1} - X_{2n-1})_{(-1)} &= D_n^- \mathcal{D}_+ \ln \tau
 \end{aligned} \tag{20}$$

where the subscript (-1) indicates taking the coefficient in front of \mathcal{D}_+^{-1} in the expansion of the corresponding super-pseudo-differential operator. The validity of Eqs.(20) follows directly from the $N=2$ superspace Zakharov-Shabat “zero-curvature” equations, *i.e.*, the compatibility conditions among the isospectral Sato

evolution equations (16) respecting the (anti-)commutation isospectral flow algebra (5).

4. Superloop Superalgebra Symmetries of Constrained $N=2$ Super-KP Hierarchies

In the study ¹¹ of additional non-isospectral symmetries of $N=1$ supersymmetric integrable hierarchies we encountered superloop superalgebras $\widehat{GL}(N_1, N_2)$. The latter are infinite-dimensional super-Lie algebras with half-integer loop grading $0, \pm\frac{1}{2}, \pm 1, \pm\frac{3}{2}, \dots$:

$$\widehat{GL}(N_1, N_2) = \bigoplus_{\ell \in \mathbb{Z}} GL^{(\frac{\ell}{2})}(N_1, N_2) \quad (21)$$

whose $\frac{\ell}{2}$ -grade subspaces consist of super-matrices of the following form:

$$GL^{(n)}(N_1, N_2) = \left\{ \begin{pmatrix} A^{(n)} & B^{(n)} \\ C^{(n)} & D^{(n)} \end{pmatrix} \in Mat(N_1, N_2) \right\} \quad (22)$$

$$GL^{(n-\frac{1}{2})}(N_1, N_2) = \left\{ \begin{pmatrix} B^{(n-\frac{1}{2})} & A^{(n-\frac{1}{2})} \\ D^{(n-\frac{1}{2})} & C^{(n-\frac{1}{2})} \end{pmatrix} \in \widetilde{Mat}(N_1, N_2) \right\} \quad (23)$$

In Eq.(22) $Mat(N_1, N_2)$ denotes the space of all $(N_1, N_2) \times (N_1, N_2)$ supermatrices defined in the standard Berezin block form (*i.e.*, diagonal blocks are bosonic and off-diagonal blocks are fermionic), whereas the symbol $\widetilde{Mat}(N_1, N_2)$ in Eq.(23) indicates the space of $(N_1, N_2) \times (N_1, N_2)$ supermatrices given in a nonstandard block format with diagonal blocks being fermionic while the off-diagonal blocks are bosonic (for a detailed discussion of non-standard formats of super-Lie algebras, see ref.¹⁶). Below we will show that the same type of superloop superalgebras serve as algebras of additional symmetry flows of $N=2$ super-KP hierarchies.

Borrowing from the construction in ref.¹¹ let us consider the following infinite sets of bosonic and fermionic super-pseudo-differential operators:

$$\mathcal{M}_{\mathcal{A}}^{(\ell/2)} \equiv \sum_{i,j=1}^M \mathcal{A}_{ij}^{(\ell/2)} \sum_{s=0}^{\ell-1} (-1)^{s(|j|+\ell)} \mathcal{L}^{\ell-1-s} (\Phi_j) \mathcal{D}_+^{-1} (\mathcal{L}^s)^* (\Psi_i) \quad (24)$$

$$\mathcal{M}_{\mathcal{F}}^{(\ell/2)} \equiv \sum_{i,j=1}^M \mathcal{F}_{ij}^{(\ell/2)} \sum_{s=0}^{\ell-1} (-1)^{s(|j|+\ell)} \mathcal{L}^{\ell-1-s} (\Phi_j) \mathcal{D}_+^{-1} (\mathcal{L}^s)^* (\Psi_i) \quad (25)$$

where $\ell = 1, 2, \dots$ and $\{\Phi_i, \Psi_i\}$ are the same SEF's entering the pseudo-differential part of \mathcal{L} (13). The associated flows generated by (24)–(25) on the $N=2$ constrained super-KP hierarchies (13) read:

$$\delta_{\mathcal{A}}^{(\ell/2)} \mathcal{L} = \left[\mathcal{M}_{\mathcal{A}}^{(\ell/2)}, \mathcal{L} \right] \quad , \quad \delta_{\mathcal{F}}^{(\ell/2)} \mathcal{L} = \left\{ \mathcal{M}_{\mathcal{F}}^{(\ell/2)}, \mathcal{L} \right\} \quad (26)$$

or, equivalently, $\delta_{\mathcal{A}, \mathcal{F}}^{(\ell/2)} \mathcal{W} = \mathcal{M}_{\mathcal{A}, \mathcal{F}}^{(\ell/2)} \mathcal{W}$. In Eqs.(24)–(25) $\mathcal{A}^{(\ell/2)}$ and $\mathcal{F}^{(\ell/2)}$ are constant supermatrices belonging to the superloop superalgebra $\widehat{GL}(M_B, M_F)$ (cf. definition (21)–(23)) of the following types:

(a) For $\ell = 2n$ $\mathcal{A}^{(n)}$ and $\mathcal{F}^{(n)}$ are purely bosonic and purely fermionic elements, respectively:

$$\mathcal{A}^{(n)} = \begin{pmatrix} A^{(n)} & 0 \\ 0 & D^{(n)} \end{pmatrix}, \quad \mathcal{F}^{(n)} = \begin{pmatrix} 0 & B^{(n)} \\ C^{(n)} & 0 \end{pmatrix} \quad (27)$$

Here the block matrices $A^{(n)}$, $B^{(n)}$, $C^{(n)}$ and $D^{(n)}$ are of sizes $M_B \times M_B$, $M_B \times M_F$, $M_F \times M_B$ and $M_F \times M_F$, respectively.

(b) For $\ell = 2n - 1$ the supermatrices $\mathcal{A}^{(n-\frac{1}{2})}$ and $\mathcal{F}^{(n-\frac{1}{2})}$ are purely bosonic and purely fermionic elements in the “twisted” non-standard format:

$$\mathcal{F}^{(n-\frac{1}{2})} = \begin{pmatrix} B^{(n-\frac{1}{2})} & 0 \\ 0 & C^{(n-\frac{1}{2})} \end{pmatrix}, \quad \mathcal{A}^{(n-\frac{1}{2})} = \begin{pmatrix} 0 & A^{(n-\frac{1}{2})} \\ D^{(n-\frac{1}{2})} & 0 \end{pmatrix} \quad (28)$$

In this case the sizes of the block matrices $A^{(n-\frac{1}{2})}$, $B^{(n-\frac{1}{2})}$, $C^{(n-\frac{1}{2})}$ and $D^{(n-\frac{1}{2})}$ are $M_B \times M_F$, $M_B \times M_B$, $M_F \times M_F$ and $M_F \times M_B$, respectively.

Note from Eq.(24) that for $\ell = 2n$:

$$\mathcal{M}_{\mathcal{A}=\mathbb{I}}^{(n)} \equiv \sum_{j=1}^M \sum_{s=0}^{2n-1} (-1)^{s|j|} \mathcal{L}^{2n-1-s}(\Phi_j) \mathcal{D}_+^{-1} (\mathcal{L}^s)^* (\Psi_j) = (\mathcal{L}^{2n})_- \quad (29)$$

whereas for $\ell = 2n - 1$ Eq.(25) implies:

$$\mathcal{M}_{\mathcal{F}=\mathbb{I}}^{(n-\frac{1}{2})} \equiv \sum_{j=1}^M \sum_{s=0}^{2n-2} (-1)^{s(|j|+1)} \mathcal{L}^{2n-2-s}(\Phi_j) \mathcal{D}_+^{-1} (\mathcal{L}^s)^* (\Psi_j) = (\mathcal{L}^{2n-1})_- - X_{2n-1} \quad (30)$$

with X_{2n-1} the same as in Eq.(17).

Since all superfunctions entering the operators $\mathcal{M}_{\mathcal{A},\mathcal{F}}^{(\ell/2)}$ are SEF's of the constrained $N = 2$ super-KP hierarchies (13), it is straightforward to show that the flows $\delta_{\mathcal{A},\mathcal{F}}^{(\ell/2)}$ (26) commute with the bosonic isospectral flows $\frac{\partial}{\partial t_i}$ (4), thus, identifying (26) as additional non-isospectral symmetries of the latter hierarchies. Furthermore, $\delta_{\mathcal{A},\mathcal{F}}^{(\ell/2)}$ -flows (anti-)commute with the second set of fermionic isospectral flows D_n^+ (4), whereas the first set of fermionic isospectral flows D_n^- (4) become nontrivial elements of an infinite-dimensional superloop superalgebra (cf. (39) and (40) below).

In checking the above properties we make use of the following super-pseudo-differential operator identities (cf. refs.^{14,11}):

$$\mathcal{Z}_{(i,j)} \mathcal{Z}_{(k,l)} = \mathcal{Z}_{(i,j)}(\Phi_k) \mathcal{D}_+^{-1} \Psi_l + (-1)^{|j|(|k|+|l|+1)} \Phi_i \mathcal{D}_+^{-1} \mathcal{Z}_{(k,l)}^* (\Psi_j) \quad (31)$$

where $\mathcal{Z}_{(i,j)} \equiv \Phi_i \mathcal{D}_+^{-1} \Psi_j$ and similarly for $\mathcal{Z}_{(k,l)}$.

Consistency of the flow actions (26) with the constrained form of the $N = 2$ super-Lax operator (13) implies, using identities (31), the following actions of $\delta_{\mathcal{A},\mathcal{F}}^{(\ell/2)}$ -flows on the constituent (adjoint) SEF's:

$$\delta_{\mathcal{A}}^{(\ell/2)} \Phi_i = \mathcal{M}_{\mathcal{A}}^{(\ell/2)}(\Phi_i) - \sum_{j=1}^M \mathcal{A}_{ij}^{(\ell/2)} \mathcal{L}^\ell(\Phi_j) \quad (32)$$

$$\delta_{\mathcal{A}}^{(\ell/2)} \Psi_i = - \left(\mathcal{M}_{\mathcal{A}}^{(\ell/2)} \right)^* (\Psi_i) + \sum_{j=1}^M (-1)^{\ell|j|} \mathcal{A}_{ji}^{(\ell/2)} (\mathcal{L}^\ell)^* (\Psi_j) \quad (33)$$

$$\delta_{\mathcal{F}}^{(\ell/2)} \Phi_i = \mathcal{M}_{\mathcal{F}}^{(\ell/2)} (\Phi_i) + \sum_{j=1}^M \mathcal{F}_{ij}^{(\ell/2)} \mathcal{L}^\ell (\Phi_j) \quad (34)$$

$$\delta_{\mathcal{F}}^{(\ell/2)} \Psi_i = - \left(\mathcal{M}_{\mathcal{F}}^{(\ell/2)} \right)^* (\Psi_i) - \sum_{j=1}^M (-1)^{(\ell+1)|j|} \mathcal{F}_{ji}^{(\ell/2)} (\mathcal{L}^\ell)^* (\Psi_j) \quad (35)$$

Furthermore, employing again identities (31), we obtain:

$$\delta_{\mathcal{A}_1}^{(\ell/2)} \mathcal{M}_{\mathcal{A}_2}^{(m/2)} - \delta_{\mathcal{A}_2}^{(m/2)} \mathcal{M}_{\mathcal{A}_1}^{(\ell/2)} - \left[\mathcal{M}_{\mathcal{A}_1}^{(\ell/2)}, \mathcal{M}_{\mathcal{A}_2}^{(m/2)} \right] = \mathcal{M}_{[\mathcal{A}_1, \mathcal{A}_2]}^{((\ell+m)/2)} \quad (36)$$

$$\delta_{\mathcal{A}}^{(\ell/2)} \mathcal{M}_{\mathcal{F}}^{(m/2)} - \delta_{\mathcal{F}}^{(m/2)} \mathcal{M}_{\mathcal{A}}^{(\ell/2)} - \left[\mathcal{M}_{\mathcal{A}}^{(\ell/2)}, \mathcal{M}_{\mathcal{F}}^{(m/2)} \right] = \begin{cases} \mathcal{M}_{[\mathcal{A}, \mathcal{F}]}^{((\ell+m)/2)} & \text{for } \ell = \text{even} \\ -\mathcal{M}_{\{\mathcal{A}, \mathcal{F}\}}^{((\ell+m)/2)} & \text{for } \ell = \text{odd} \end{cases} \quad (37)$$

$$\begin{aligned} & \delta_{\mathcal{F}_1}^{(\ell/2)} \mathcal{M}_{\mathcal{F}_2}^{(m/2)} + \delta_{\mathcal{F}_2}^{(m/2)} \mathcal{M}_{\mathcal{F}_1}^{(\ell/2)} - \left\{ \mathcal{M}_{\mathcal{F}_1}^{(\ell/2)}, \mathcal{M}_{\mathcal{F}_2}^{(m/2)} \right\} = \\ & = \begin{cases} \pm \mathcal{M}_{\{\mathcal{F}_1, \mathcal{F}_2\}}^{((\ell+m)/2)} & \text{for } (\ell, m) = (\text{odd}, \text{odd}) / (\text{even}, \text{even}) \\ \pm \mathcal{M}_{[\mathcal{F}_1, \mathcal{F}_2]}^{((\ell+m)/2)} & \text{for } (\ell, m) = (\text{odd}, \text{even}) / (\text{even}, \text{odd}) \end{cases} \end{aligned} \quad (38)$$

The super-pseudo-differential operator relations (36)–(38), which constitute the compatibility conditions for the additional symmetry flows (26), imply that the latter obey the following infinite-dimensional algebra (here $\ell, m \geq 1$):

$$\begin{aligned} \left[\delta_{\mathcal{A}_1}^{(\ell/2)}, \delta_{\mathcal{A}_2}^{(m/2)} \right] &= \delta_{[\mathcal{A}_1, \mathcal{A}_2]}^{((\ell+m)/2)}, \quad \left[\delta_{\mathcal{A}}^{(\ell/2)}, \delta_{\mathcal{F}}^{(m/2)} \right] = \delta_{[\mathcal{A}, \mathcal{F}]}^{((\ell+m)/2)} \quad \text{for } \ell = \text{even} \\ \left[\delta_{\mathcal{A}}^{(\ell/2)}, \delta_{\mathcal{F}}^{(m/2)} \right] &= -\delta_{\{\mathcal{A}, \mathcal{F}\}}^{((\ell+m)/2)} \quad \text{for } \ell = \text{odd} \\ \left\{ \delta_{\mathcal{F}_1}^{(\ell/2)}, \delta_{\mathcal{F}_2}^{(m/2)} \right\} &= \pm \delta_{\{\mathcal{F}_1, \mathcal{F}_2\}}^{((\ell+m)/2)} \quad \text{for } (\ell, m) = (\text{odd}, \text{odd}) / (\text{even}, \text{even}) \\ \left\{ \delta_{\mathcal{F}_1}^{(\ell/2)}, \delta_{\mathcal{F}_2}^{(m/2)} \right\} &= \pm \delta_{[\mathcal{F}_1, \mathcal{F}_2]}^{((\ell+m)/2)} \quad \text{for } (\ell, m) = (\text{odd}, \text{even}) / (\text{even}, \text{odd}) \end{aligned} \quad (39)$$

The algebra (39) is isomorphic to the positive-grade subalgebra of the superloop superalgebra $\widehat{GL}(M_B, M_F)$. In particular, from (29)–(30) we find that:

$$\delta_{\mathcal{A}=\mathbb{1}}^{(n)} = -\frac{\partial}{\partial t_n}, \quad \delta_{\mathcal{F}=\mathbb{1}}^{(n-\frac{1}{2})} = -D_n^- \quad (40)$$

are, upto an overall minus sign, isospectral flows of the corresponding $SKP_{(M_B, M_F)}^{N=2}$ hierarchy (13).

5. Darboux-Bäcklund (N=2 “Super-Soliton”) Solutions

In refs.^{10,11} we have discussed in detail Darboux-Bäcklund (DB) transformations and DB solutions (*i.e.*, soliton-like solutions) of bosonic and $N=1$ supersymmetric constrained KP hierarchies preserving the additional non-isospectral symmetries of the latter. Employing the same approach we are lead to consider the following DB and adjoint-DB transformations of $N=2$ constrained super-KP hierarchies $SKP_{(M_B, M_F)}^{N=2}$ (13) :

$$\tilde{\mathcal{L}} = -\mathcal{T}_\phi \mathcal{L} \mathcal{T}_\phi^{-1} \quad , \quad \mathcal{T}_\phi \equiv \phi \mathcal{D}_+ \phi^{-1} \quad , \quad \phi \equiv \Phi_{i_0} \quad (41)$$

$$\hat{\mathcal{L}} = (\mathcal{T}_\psi^{-1})^* \mathcal{L} \mathcal{T}_\psi^* \quad , \quad \mathcal{T}_\psi \equiv \psi \mathcal{D}_+ \psi^{-1} \quad , \quad \psi \equiv \Psi_{i_0} \quad (42)$$

where Φ_{i_0} and Ψ_{i_0} are some fixed bosonic (adjoint) SEF's entering the pseudo-differential part of the $N=2$ super-Lax operator (13). The (adjoint) DB-transformed super-Lax operator is of the same form as the initial one:

$$\tilde{\mathcal{L}} = \mathcal{D}_- + \sum_{i=1}^M \tilde{\Phi}_i \mathcal{D}_+^{-1} \tilde{\Psi}_i \quad , \quad \hat{\mathcal{L}} = \mathcal{D}_- + \sum_{i=1}^M \hat{\Phi}_i \mathcal{D}_+^{-1} \hat{\Psi}_i \quad (43)$$

$$\tilde{\Phi}_{i_0} = -\mathcal{T}_\phi (\mathcal{L}(\Phi_{i_0})) \quad , \quad \tilde{\Psi}_{i_0} = \frac{1}{\Phi_{i_0}} \quad ; \quad \hat{\Phi}_{i_0} = -\frac{1}{\Psi_{i_0}} \quad , \quad \hat{\Psi}_{i_0} = \mathcal{T}_\psi (\mathcal{L}^*(\Psi_{i_0})) \quad (44)$$

$$\tilde{\Phi}_i = -\mathcal{T}_\phi(\Phi_i) \quad , \quad \tilde{\Psi}_i = (-1)^{|i|} (\mathcal{T}_\phi^{-1})^*(\Psi_i) \quad \text{for } i \neq i_0 \quad (45)$$

$$\hat{\Phi}_i = (-1)^{|i|} (\mathcal{T}_\psi^{-1})^*(\Phi_i) \quad , \quad \hat{\Psi}_i = \mathcal{T}_\psi(\Psi_i) \quad \text{for } i \neq i_0 \quad (46)$$

(recall $\phi \equiv \Phi_{i_0}$, $\psi \equiv \Psi_{i_0}$). Note that the (adjoint) DB-transformed SEF's for all $i \neq i_0$ possess opposite Grassmann parities w.r.t. the initial SEF's. Further we can show, following the method of ref.¹¹, that the super-DB transformations (41)–(46) preserve (upto an overall sign change of the fermionic flows) the subalgebra $(\widehat{GL}(M_B - 1, M_F))_+$ of the additional symmetry algebra $(\widehat{GL}(M_B, M_F))_+$ (39).

Using relations (20) we deduce the following (adjoint) DB transformations for the pertinent $N=2$ super-tau function:

$$\tau \longrightarrow \tilde{\tau} = \frac{\phi}{\tau} \quad , \quad \tau \longrightarrow \hat{\tau} = -\frac{1}{\psi \tau} \quad (47)$$

Relations (47) are similar to those in the $N=1$ super-KP case^{14,11} and have to be contrasted with their counterparts in the ordinary “bosonic” case where $\tilde{\tau} = \phi \tau$, $\hat{\tau} = -\psi \tau$.

Arbitrary iterations of any number of successive DB and adjoint-DB transformations, *i.e.*, general DB orbits of $N=1$ constrained super-KP hierarchies have been worked out in detail in ref.¹¹. The current $N=2$ super-DB transformations (43)–(46) and (47) have a structure analogous to their counterparts in the $N=1$ super-KP case, with the following simple modifications: (i) the $N=1$ super-derivative operator $\mathcal{D} = \partial/\partial\theta + \theta\partial$ is replaced by the first $N=2$ super-derivative operator \mathcal{D}_+ ; (ii) the $N=1$ super-Lax operator is replaced by the $N=2$ super-Lax

operator containing the second $N=2$ super-derivative operator \mathcal{D}_- in its positive differential part (cf. (13)). Therefore, we can easily generalize the results in ¹¹ to obtain for the general DB-orbit of the $N=2$ super-tau function the following Berezinian (super-determinant) expressions:

$$\frac{\tau^{(0;0)}}{\tau^{(n+2m;n)}} = (-1)^{mn+n(n-1)/2} \times \text{Ber} \left(\begin{array}{c|c} \widetilde{W}_{n+m,n+m}^{(n+m;m)} [\{\varphi\} | \{\psi\}] & \widetilde{W}_{m,n+m}^{(m;n)} [\{\varphi_{(\frac{1}{2})}\} | \{\psi\}] \\ \hline W_{n+m,m} [\mathcal{D}_+\varphi_0, \dots, \mathcal{D}_+\varphi_{n+m-1}] & W_{m,m} [\mathcal{D}_+\varphi_{\frac{1}{2}}, \dots, \mathcal{D}_+\varphi_{m-\frac{1}{2}}] \end{array} \right) \quad (48)$$

$$\tau^{(n+2m+1;n)} \tau^{(0;0)} = (-1)^{mn+n(n-1)/2} \times \text{Ber} \left(\begin{array}{c|c} \widetilde{W}_{n+m+1,n+m+1}^{(n+m+1;n)} [\{\varphi\}, \varphi_{n+m} | \{\psi\}] & \widetilde{W}_{m,n+m+1}^{(m;n)} [\{\varphi_{(\frac{1}{2})}\} | \{\psi\}] \\ \hline W_{n+m+1,m} [\mathcal{D}_+\varphi_0, \dots, \mathcal{D}_+\varphi_{n+m}] & W_{m,m} [\mathcal{D}_+\varphi_{\frac{1}{2}}, \dots, \mathcal{D}_+\varphi_{m-\frac{1}{2}}] \end{array} \right) \quad (49)$$

In Eqs.(48)–(49) the notations used are as follows. The superscript $(k;l)$ of the super-tau function $\tau^{(k;l)}$ indicates k steps of DB transformations plus l steps of adjoint-DB transformations according to (43)–(46) and (47). The block matrices entering the Berezinians in (48)–(49) possess the following special generalized Wronskian-like $k \times (m+n)$ matrix form:

$$\begin{aligned} \widetilde{W}_{k,m+n}^{(k;n)} [\{\varphi\} | \{\psi\}] &\equiv \widetilde{W}_{k,m+n}^{(k;n)} [\varphi_0, \dots, \varphi_{k-1} | \psi_{\frac{1}{2}}, \dots, \psi_{n-\frac{1}{2}}] = \\ &= \begin{pmatrix} \varphi_0 & \cdots & \cdots & \varphi_{k-1} \\ \vdots & \ddots & \ddots & \vdots \\ \partial^{m-1} \varphi_0 & \cdots & \cdots & \partial^{m-1} \varphi_{k-1} \\ \mathcal{D}_+^{-1}(\varphi_0 \psi_{\frac{1}{2}}) & \cdots & \cdots & \mathcal{D}_+^{-1}(\varphi_{k-1} \psi_{\frac{1}{2}}) \\ \vdots & \ddots & \ddots & \vdots \\ \mathcal{D}_+^{-1}(\varphi_0 \psi_{n-\frac{1}{2}}) & \cdots & \cdots & \mathcal{D}_+^{-1}(\varphi_{k-1} \psi_{n-\frac{1}{2}}) \end{pmatrix} \end{aligned} \quad (50)$$

where $\{\varphi\} \equiv \{\varphi_0, \dots, \varphi_{k-1}\}$ is a set of k bosonic or fermionic superfunctions whereas $\{\psi\} \equiv \{\psi_{\frac{1}{2}}, \dots, \psi_{n-\frac{1}{2}}\}$ is a set of n fermionic superfunctions. The generalized Wronskian-like matrix (50) is the $N=2$ supersymmetric generalization of the Wronskian-like block matrices entering the general Darboux-Bäcklund determinant solutions for the tau-functions of ordinary “bosonic” constrained KP hierarchies ^{8,10}. In the special case of $n=0$ (50) reduces to the rectangular $k \times m$ Wronskian

matrix:

$$W_{k,m}[\varphi_0, \dots, \varphi_{k-1}] = \begin{pmatrix} \varphi_0 & \cdots & \cdots & \varphi_{k-1} \\ \partial\varphi_0 & \cdots & \cdots & \partial\varphi_{k-1} \\ \vdots & \ddots & \ddots & \vdots \\ \partial^{m-1}\varphi_0 & \cdots & \cdots & \partial^{m-1}\varphi_{k-1} \end{pmatrix} \quad (51)$$

Accordingly, the sets of superfunctions entering the Wronskian-like matrix blocks in the Berezinians (48)–(49) are bosonic SEF's ($\{\varphi\} \equiv \{\varphi_0, \dots, \varphi_{n+m-1}\}$), fermionic SEF's ($\{\varphi_{(\frac{1}{2})}\} \equiv \{\varphi_{\frac{1}{2}}, \dots, \varphi_{m-\frac{1}{2}}\}$) and fermionic adjoint SEF's ($\{\psi\} \equiv \{\psi_{\frac{1}{2}}, \dots, \psi_{n-\frac{1}{2}}\}$), respectively, which are expressed in terms of the constituent (adjoint) SEF's $\{\Phi_i, \Psi_i\}_{i=1}^M$ entering the pseudo-differential part of the super-Lax operator (13) in the following way:

$$\{\varphi\} = \left(\left\{ \mathcal{L}^{2p_k}(\Phi_k) \right\}_{k=1, \dots, M_B}^{p_k=0, 1, \dots}, \left\{ \mathcal{L}^{2q_l+1}(\Phi_{M_B+l}) \right\}_{l=1, \dots, M_F}^{q_l=0, 1, \dots} \right) \quad (52)$$

$$\{\varphi_{(\frac{1}{2})}\} = \left(\left\{ \mathcal{L}^{2r_k}(\Phi_{M_B+k}) \right\}_{k=1, \dots, M_F}^{r_k=0, 1, \dots}, \left\{ \mathcal{L}^{2s_l+1}(\Phi_l) \right\}_{l=1, \dots, M_B}^{s_l=0, 1, \dots} \right) \quad (53)$$

$$\{\psi\} = \left(\left\{ (\mathcal{L}^{2r_k})^*(\Psi_{M_B+k}) \right\}_{k=1, \dots, M_F}^{r_k=0, 1, \dots}, \left\{ (\mathcal{L}^{2s_l+1})^*(\Psi_l) \right\}_{l=1, \dots, M_B}^{s_l=0, 1, \dots} \right) \quad (54)$$

(recall that $\{\Phi_i, \Psi_i\}_{i=1}^{M_B}$ are bosonic, whereas $\{\Phi_{M_B+i}, \Psi_{M_B+i}\}_{i=1}^{M_F}$ are fermionic).

In conclusion let us write down explicitly the general DB-solution for the $N=2$ super-tau function (48)–(49) of the simplest member of constrained $N=2$ super-KP hierarchies' class (13) with $M=1$, i.e., $\mathcal{L} = \mathcal{D}_- + \Phi \mathcal{D}_+^{-1} \Psi$ with “free” initial super-Lax operator $\mathcal{L}^{(0;0)} = \mathcal{D}_-$. It turns out that the latter solutions coincide with the DB-solutions previously obtained in refs.^{12,17}.

Indeed, in this case one can show, taking into account (52)–(53), that the Berezinian expressions (48)–(49) reduce to ‡

$$(\tau^{(2m;0)})^{-1} = \quad (55)$$

$$\text{Ber} \left(\begin{array}{c|c} W_{m,m}[\Phi_0, \dots, \partial^{m-1}\Phi_0] & W_{m,m}[\mathcal{D}_-\Phi_0, \dots, \mathcal{D}_-\partial^{m-1}\Phi_0] \\ \hline W_{m,m}[\mathcal{D}_+\Phi_0, \dots, \mathcal{D}_+\partial^{m-1}\Phi_0] & W_{m,m}[\mathcal{D}_+\mathcal{D}_-\Phi_0, \dots, \mathcal{D}_+\mathcal{D}_-\partial^{m-1}\Phi_0] \end{array} \right) \quad (56)$$

$$\tau^{(2m+1;0)} =$$

$$\text{Ber} \left(\begin{array}{c|c} W_{m+1,m+1}[\Phi_0, \dots, \partial^m\Phi_0] & W_{m,m+1}[\mathcal{D}_-\Phi_0, \dots, \mathcal{D}_-\partial^{m-1}\Phi_0] \\ \hline W_{m+1,m}[\mathcal{D}_+\Phi_0, \dots, \mathcal{D}_+\partial^m\Phi_0] & W_{m,m}[\mathcal{D}_+\mathcal{D}_-\Phi_0, \dots, \mathcal{D}_+\mathcal{D}_-\partial^{m-1}\Phi_0] \end{array} \right)$$

‡In this simplest case of $N=2$ super-KP hierarchy, similarly to its bosonic and $N=1$ super-KP counterparts, adjoint-DB transformations correspond simply to a backward shift on the DB orbit.

where notation (51) for the pertinent Wronskian matrix blocks has been used. In Eqs.(55)–(56) Φ_0 denotes the “free” SEF (cf. (12) and (7)) :

$$\Phi_0(t, \theta_{\pm}, \rho^{\pm}) = \int d\lambda d\eta_{\pm} \phi_0(\lambda, \eta_{\pm}) e^{\xi(t, \theta_{\pm}, \pm \rho^{\pm}; \lambda, \eta_{\pm})} \quad (57)$$

$$\phi_0(\lambda, \eta_{\pm}) = \phi_B^{(1)}(\lambda) + \eta_+ \phi_F^{(1)}(\lambda) + \eta_- (\phi_F^{(2)}(\lambda) + \eta_+ \phi_B^{(2)}(\lambda)) \quad (58)$$

with an arbitrary $N = 2$ superspace “density” (58). As in the bosonic and $N = 1$ cases, supersymmetric “soliton”-like solutions for the $N = 2$ super-tau function (55)–(56) are obtained by choosing delta-function form for the components of the $N = 2$ superspace “density” (58) :

$$\phi_B^{(1,2)} = \sum_{a=1}^{N_B} c_a^{(1,2)} \delta(\lambda - \lambda_a) \quad , \quad \phi_F^{(1,2)} = \sum_{a=1}^{N_F} \varepsilon_a^{(1,2)} \delta(\lambda - \lambda_a) \quad (59)$$

with arbitrary bosonic $\{c_a^{(1,2)}, \lambda_a\}$ and fermionic $\{\varepsilon_a^{(1,2)}\}$ constant parameters, and $N_{B,F}$ being arbitrary integers.

Outlook

The present results open a number of interesting problems for further research, in particular:

(i) Search for a negative-grade counterpart of the positive-grade superloop superalgebra $\left(\widehat{GL}(M_B, M_F)\right)_+$ (39) of additional symmetries of $N = 2$ (constrained) super-KP hierarchies (13), as it is the case for bosonic and $N = 1$ super-KP hierarchies^{9,10,11}.

(ii) Search for Virasoro (conformal) and super-Virasoro additional symmetries of $N = 2$ super-KP hierarchies (13) (for the derivation of Virasoro additional symmetries of constrained bosonic and $N = 1$ super-KP hierarchies, see refs.^{9,11}).

(iii) Detailed study of *extended* $N = 2$ super-KP hierarchies, obtained from the $SKP_{(M_B, M_F)}^{N=2}$ hierarchies (13) by adding to the set of original isospectral flows $\frac{\partial}{\partial t_i}$, D_n^{\pm} , additional Manin-Radul-type subsets of (anti-)commuting additional symmetry flows, as it has been done in the $N = 1$ super-KP case¹¹. Similarly to the bosonic and $N = 1$ super-KP cases (see refs.^{8,10,11}) we can view the above extended $N = 2$ super-KP hierarchies as *matrix* generalizations of the original scalar $SKP_{(M_B, M_F)}^{N=2}$ hierarchies (13). Furthermore, as in the $N = 1$ super-KP case we can look for $N = 2$ supersymmetric generalizations of various interesting higher-dimensional nonlinear evolution equations (such as Davey-Stewartson nonlinear system) which are contained in the extended (matrix) $N = 2$ super-KP hierarchies.

(iv) Detailed study of the properties and possible physical significance of the very broad class of Darboux-Bäcklund solutions (48)–(49), in particular (55)–(56).

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